JOURNAL OF APPROXIMATION THEORY 71, 123–137 (1992)

Interpolation of Entire Functions Associated with Some Freud Weights, I

RADWAN AL-JARRAH*

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210, U.S.A.

AND

MOHAMED R. HASAN

Department of Mathematics, Yarmouk University, Irbid, Jordan

Communicated by Doron S. Lubinsky

Received February 4, 1991; accepted October 10, 1991

We investigate in this paper the geometric convergence of the Lagrange and Hermite interpolation processes and the Gauss–Jacobi quadrature formula of an entire function and its higher derivatives when the nodes of interpolation are taken to be the zeros of the orthogonal polynomials associated with the very smooth Freud weight

$$w_\alpha(x) = \exp(-2|x|^\alpha), \qquad \alpha > 0,\ x \in \mathbb{R}.$$

In each of these approximations, we give a numerical bound on the growth of the function and we estimate the corresponding error term.

1. MOTIVATION

Let W be the class of the Freud weights of the form $w_Q(x) = \exp\{-2Q(x)\}$, $x \in \mathbb{R}$, where

(i) $Q(x)$ is an even, differentiable function, except possibly at $x = 0$, increasing for $x > 0$,

(ii) there exists $\rho < 1$ such that $x^\rho Q'(x)$ is increasing, and

(iii) the sequence $\{q_n\}$ determined by $q_s Q'(q_s) = s$ satisfies the condition $q_{2n}/q_n \geqslant c > 1$, $n = 1, 2, 3, \ldots$, for some constant c independent of n.

* Present address: Department of Mathematics, Southwestern Oklahoma State University, Weatherford, OK 73096.

0021-9045/92 $5.00

Observe that whenever $Q(x) = Q_\alpha(x) = |x|^\alpha$, $\alpha \geqslant 1$, then

$$w_Q(x) = w_\alpha(x) = \exp(-2|x|^\alpha) \in W.$$

Also, for an entire function f, let

$$K = \limsup_{R \to \infty} \frac{\max_{|z|=R} (\log|f(z)|)}{2Q(R)}, \qquad z \in \mathbb{C},$$

and let $Q_n(w_Q; f)$ be the Gauss–Jacobi quadrature formula of f based on the zeros of the orthogonal polynomials associated with the weights w_Q of the class W.

In 1980, Al-Jarrah [1] proved the following theorem. There exists a constant $A \in (0, 1)$, depending on Q only, such that if $K \leqslant A$, then

$$\limsup_{n \to \infty} \left| \int_{\mathbb{R}} f(x) \exp(-2Q(x))\, dx - Q_n(w_Q; f) \right|^{1/n} < 1.$$

In 1983, Lubinsky [11] investigated geometric convergence for rules of numerical integration and the associated Lagrange interpolation polynomials over unbounded contours and intervals. Lubinsky's results contain, among other things, Al-Jarrah's results of [1]. He also removed restriction (iii) on the class W (see [11; Theorem 5.1]). Later on, Al-Jarrah [4, 5] obtained some results similar to those in [1] concerning the Lagrange and the Hermite interpolation processes. Neither Al-Jarrah nor Lubinsky gave a numerical value to the constant A. However, Al-Jarrah [2–5] investigated the values of A for the special weights $w_\alpha(x) = \exp(-2|x|^\alpha)$, $x \in \mathbb{R}$, $\alpha = 2, 4$, and 6, using the validity of Freud's conjecture for these weights.

In this paper, we investigate the values of A, not necessarily best possible, for the class of Freud weights $w_\alpha(x) = \exp(-2|x|^\alpha)$, $x \in \mathbb{R}$, $\alpha > 0$. This investigation was made possible after the proof of Freud's conjecture for exponential weights was published by Lubinsky, Mhaskar, and Saff [12, 13] in a more general form.

Before closing this section, we point out that, among others, the papers of Goncar and Rahmanov [9], Knopfnacher and Lubinsky [10], Mhaskar [14, 15], Mhaskar and Saff [16], and Nevai [17] are also close in spirit to the subject matter of this paper.

2. Introduction

Given the very smooth Freud weight function

$$w_\alpha(x) = \exp(-2|x|^\alpha), \qquad \alpha > 0,\ x \in \mathbb{R},$$

it is well known that there exists a unique sequence of orthonormal polynomials $\{p_n(w_\alpha; x)\}$ (see [6, I.1]) associated with $w_\alpha(x)$ with the properties:

(i) $p_n(w_\alpha; x) = \gamma_n x^n + \cdots$ is a polynomial of degree n and $\gamma_n > 0$;
(ii) $\int_{\mathbb{R}} p_n(w_\alpha; x)\, p_m(w_\alpha; x)\, w_\alpha(x)\, dx = \delta_{nm}$, the Kronecker delta.

The zeros $\{x_{kn}\}_{k=1}^{n}$ of $p_n(w_\alpha; x)$ are all real and simple. We assume, as usual, that $x_{1n} > x_{2n} > \cdots > x_{nn}$.

For a given function f on $\mathbb{R}$, the Lagrange interpolation polynomial $L_n(w_\alpha; f)$ associated with $w_\alpha(x)$ is defined to be the unique polynomial of degree at most $n-1$ which coincides with f at the nodes x_{kn}. In fact,

$$L_n(w_\alpha; f; x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x),$$

where $l_{kn}(x)$ are the fundamental polynomials of Lagrange interpolation defined by

$$l_{kn}(x) = \frac{p_n(w_\alpha; x)}{p_n'(w_\alpha; x_{kn})(x - x_{kn})}, \qquad (k = 1, 2, 3, ..., n).$$

If, in addition, the function f is differentiable, then the Hermite interpolation polynomial $H_n(w_\alpha; f)$ associated with $w_\alpha(x)$ is defined to be the unique polynomial of degree $2n-1$ at most which satisfies

$$H_n(w_\alpha; f; x_{kn}) = f(x_{kn}), \qquad H_n'(w_\alpha; f; x_{kn}) = f'(x_{kn}), \qquad (k = 1, 2, 3, ..., n).$$

The Gauss–Jacobi quadrature formula to the function f that is associated with $w_\alpha(x)$ is defined by

$$Q_n(w_\alpha; f) = \sum_{k=1}^{n} \lambda_n(w_\alpha; x_{kn})\, f(x_{kn}) \qquad \left(\sim \int_{\mathbb{R}} f(x)\, w_\alpha(x)\, dx\right),$$

where the coefficients $\lambda_n(w_\alpha; x_{kn})$ are the Christoffel numbers and are given by

$$\lambda_n^{-1}(w_\alpha; x) = \sum_{k=1}^{n-1} p_k^2(w_\alpha; x),$$

and the nodes x_{kn} are the Gaussian abscissas with respect to $w_\alpha(x)$.

If f is an entire function, then the error terms when approximating f by $L_n(w_\alpha; f)$, $H_n(w_\alpha; f)$, and $Q_n(w_\alpha; f)$ are given by

$$f(\xi) - L_n(w_\alpha; f; \xi) = \frac{p_n(w_\alpha; \xi)}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{p_n(w_\alpha; z)(z-\xi)}, \tag{2.1}$$

$$f(\xi) - H_n(w_\alpha; f; \xi) = \frac{p_n^2(w_\alpha; \xi)}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{p_n^2(w_\alpha; z)(z-\xi)}, \tag{2.2}$$

and

$$\int_{\mathbb{R}} f(x)\, w_\alpha(x)\, dx - Q_n(w_\alpha; f)$$

$$= \sum_{k=n}^{\infty} \frac{\gamma_{k+1}}{\gamma_k} \cdot \frac{1}{2\pi i} \oint_{C_k} \frac{f(z)\,dz}{p_k(w_\alpha; z)\, p_{k+1}(w_\alpha, z)}, \tag{2.3}$$

respectively, where $\xi \in \mathbb{C}$, $C_k \subseteq \mathscr{D} \subseteq \mathbb{C}$, and $\mathscr{D}$ is a simply connected domain containing the zeros of $p_k(w_\alpha; x)$ in its interior. For more on (2.1) and (2.2) see [6, III, 8.4], and for more on (2.3) see [7].

3. Main Results

Throughout the rest of this paper, f will be an entire function, and $w_\alpha(x) = \exp(-2|x|^\alpha)$, $\alpha > 0$, $x \in \mathbb{R}$. To simplify the statement of our main results, we first introduce some notations. Let a be the positive solution of

$$g(x) = \frac{(1-x)}{4} \exp\left(\frac{1-x}{2x}\right) = 1, \quad (a \approx 0.23),$$

and let

$$\tau(\alpha) = \frac{(1-a)^{(\alpha+2)/2}}{2(\alpha+2)\, \beta_\alpha^\alpha a}, \qquad \text{where} \quad \beta_\alpha = \left[\frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\alpha/2)\}^2}\right]^{-1/\alpha}$$

For more on the function $\tau(\alpha)$, see the appendix at the end of the paper.

Finally, we let

$$\Delta_n^{(\alpha,m)} = \int_{\mathbb{R}} f^{(m)}(x)\, w_\alpha(x)\, dx - Q_n(w_\alpha; f^{(m)}), \qquad (m = 0, 1, 2, ...)$$

and

$$\sigma = \limsup_{R \to \infty} \frac{\log M_f(R)}{2R^\alpha}, \qquad \text{where} \quad M_f(R) = \max_{|z|=R} |f(z)|,\ z \in \mathbb{C}.$$

We now formulate our main results.

THEOREM 3.1. *If $\sigma < \tau(\alpha)$, then*

$$\limsup_{n \to \infty} (|\Delta_n^{(\alpha,m)}|)^{1/n} < 1,$$

for all $m = 0, 1, 2, \ldots$.

THEOREM 3.2. *If $\sigma < \frac{1}{2}\tau(\alpha)$, then for any $\xi \in \mathbb{C}$, we have*

$$\limsup_{n \to \infty} (|f^{(m)}(\xi) - L_n(w_\alpha; f^{(m)}; \xi)|)^{1/n} < 1,$$

for all $m = 0, 1, 2, \ldots$.

THEOREM 3.3. *If $\sigma < \tau(\alpha)$, then for any $\xi \in \mathbb{C}$, we have*

$$\limsup_{n \to \infty} (|f^{(m)}(\xi) - H_n(w_\alpha; f^{(m)}; \xi)|)^{1/n} < 1,$$

for all $m = 0, 1, 2, \ldots$.

Moreover, the last two theorems hold uniformly on compact subsets of the complex plane.

4. PRELIMINARY RESULTS

The following preliminaries are needed for the proof of our main results:

LEMMA 4.1. *For the weight function $w_\alpha(x)$, we have*

$$\max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k} \leqslant x_{1n} \leqslant 2 \max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k}, \tag{4.1}$$

$$\sum_{k=1}^{[n/2]} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2, \tag{4.2}$$

$$\lim_{n \to \infty} n^{-1/\alpha} \cdot \frac{\gamma_{n-1}}{\gamma_n} = \frac{\beta_\alpha}{2}, \tag{4.3}$$

and for all $z \in \mathbb{C}$ with $|z| \leqslant x_{1n}$, we have

$$|p_n(w_\alpha; z)| \leqslant 2^{n/2} \gamma_n x_{1n}^n. \tag{4.4}$$

For the proof of (4.1), (4.2), (4.3), and (4.4), see [8], [2], [13], and [4], respectively. In fact, (4.3) is a more generalized form of Freud's conjecture for exponential weights, which was formulated only for the case when α is a positive even integer.

LEMMA 4.2.

$$\limsup_{R \to \infty} \frac{M_{f^{(m)}}(R)}{2R^{\alpha}} \leqslant \sigma,$$

for all $m = 1, 2, 3, \ldots$.

The proof of this lemma can be easily extracted from [5, Lemma 5].

The remaining two lemmas are based on the weight function $w_\alpha(x)$. The first one is a generalization of [3, Lemma 3.6].

LEMMA 4.3. *For any* $\eta > 0$, *there exists* $N_\eta \in \mathbb{N}$, *such that, for all* $n \geqslant N_\eta$, *we have*

$$x_{1,n+1} \leqslant 2\left(\frac{\beta_\alpha}{2} + \eta\right) n^{1/\alpha}, \tag{4.5}$$

$$\sum_{k=1}^{[n/2]} x_{kn}^2 \leqslant K(\alpha, \eta) + \left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2} + \eta\right)^2 n^{(\alpha+2)/\alpha}, \tag{4.6}$$

and

$$\frac{1}{\gamma_n^2} \leqslant A(\alpha, \eta)\left(\frac{\beta_\alpha}{2} + \eta\right)^{2n} (n!)^{2/\alpha}, \tag{4.7}$$

where $K(\alpha, \eta)$ *and* $A(\alpha, \eta)$ *are two positive constants that depend on* α *and* η *only.*

Proof. Choose $\eta > 0$. Then from (4.3), it follows that there exists an $N_\eta \in \mathbb{N}$ such that, for all $n \geqslant N_\eta$, we have

$$\frac{\gamma_{n-1}}{\gamma_n} \leqslant \left(\frac{\beta_\alpha}{2} + \eta\right) n^{1/\alpha}. \tag{4.8}$$

From (4.1) and (4.8), we get

$$x_{1,n+1} \leqslant 2 \max_{1 \leqslant k \leqslant n} \frac{\gamma_{k-1}}{\gamma_k} \leqslant 2\left(\frac{\beta_\alpha}{2} + \eta\right) n^{1/\alpha},$$

which proves (4.5).

From (4.2), it follows that, for large enough n,

$$\sum_{k=1}^{[n/2]} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 = \sum_{k=1}^{N_\eta - 1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 + \sum_{k=N_\eta}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2. \tag{4.9}$$

By using (4.1), we have

$$\max_{1\leqslant k\leqslant N_\eta-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)\leqslant x_{1,N_\eta}$$

and so

$$\sum_{k=1}^{N_\eta-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2\leqslant\sum_{k=1}^{N_\eta-1}x_{1N_\eta}^2=(N_\eta-1)\,x_{1N_\eta}^2.$$

From (4.8), we conclude that

$$\begin{aligned}\sum_{k=N_\eta}^{n-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2&\leqslant\left(\frac{\beta_\alpha}{2}+\eta\right)^2\sum_{k=N_\eta}^{n-1}k^{2/\alpha}\\&\leqslant\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2}+\eta\right)^2\{(n-1)^{(\alpha+2)/\alpha}-N_\eta^{(\alpha+2)/\alpha}\}\\&\leqslant\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2}+\eta\right)^2\{n^{(\alpha+2)/\alpha}+N_\eta^{(\alpha+2)/\alpha}\}.\end{aligned}$$

Therefore, by using (4.9) and the last two inequalities, we conclude that

$$\sum_{k=1}^{[n/2]}x_{kn}^2\leqslant K(\alpha,\eta)+\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2}+\eta\right)^2n^{(\alpha+2)/\alpha},$$

where

$$K(\alpha,\eta)=(N_\eta-1)x_{1N_\eta}^2+\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2}+\eta\right)^2N_\eta^{(\alpha+2)/\alpha},$$

which proves (4.6).

Finally, to prove (4.7), we start with (4.8) to get

$$\begin{aligned}\frac{1}{\gamma_n}&\leqslant\frac{1}{\gamma_{n-1}}\left(\frac{\beta_\alpha}{2}+\eta\right)n^{1/\alpha}\\&\leqslant\frac{1}{\gamma_{n-2}}\left(\frac{\beta_\alpha}{2}+\eta\right)^2[n(n-1)]^{1/\alpha}\\&\;\;\vdots\\&\leqslant\frac{1}{\gamma_{N_\eta-1}}\left(\frac{\beta_\alpha}{2}+\eta\right)^{n-N_\eta+1}[n(n-1)\cdots(N_\eta+1)\,N_\eta]^{1/\alpha}\\&\leqslant\frac{1}{\gamma_{N_\eta-1}}\cdot\left(\frac{\beta_\alpha}{2}+\eta\right)^n(n!)^{1/\alpha}\Big/\left(\frac{\beta_\alpha}{2}+\eta\right)^{(N_\eta-1)}[(N_\eta-1)!]^{1/\alpha},\end{aligned}\tag{4.10}$$

for all $n\geqslant N_\eta$.

Next, from (4.1) we have

$$\max_{1 \leqslant k \leqslant N_\eta - 1} \left(\frac{\gamma_{k-1}}{\gamma_k} \right) \leqslant x_{1N_\eta},$$

which gives

$$\frac{1}{\gamma_k} \leqslant \frac{1}{\gamma_{k-1}} \cdot x_{1N_\eta}, \qquad \text{for all} \quad k = 1, 2, 3, ..., N_\eta - 1.$$

Therefore,

$$\frac{1}{\gamma_{N_\eta - 1}} \leqslant \frac{1}{\gamma_{N_\eta - 2}} x_{1N_\eta} \leqslant \frac{1}{\gamma_{N_\eta - 3}} x_{1N_\eta}^2 \leqslant \cdots \leqslant \frac{1}{\gamma_0} x_{1N_\eta}^{(N_\eta - 1)}.$$

By combining this inequality with (4.10), we can easily see that

$$\frac{1}{\gamma_n^2} \leqslant A(\alpha, \eta) \left(\frac{\beta_\alpha}{2} + \eta \right)^{2n} (n!)^{2/\alpha},$$

where

$$A(\alpha, \eta) = \frac{1}{\gamma_0^2} \cdot x_{1N_\eta}^{2(N_\eta - 1)} \Big/ \left(\frac{\beta_\alpha}{2} + \eta \right)^{2(N_\eta - 1)} [(N_\eta - 1)!]^{2/\alpha},$$

which completes the proof of the lemma.

LEMMA 4.4. *For all* $z \in \mathbb{C}$, *with* $|z| > x_{1n}$, *we have*

$$|p_n(w; z)|^{-1} \leqslant \frac{1}{\gamma_n |z|^n} \times \exp \left\{ \frac{K(\alpha, \eta) + (\alpha/(\alpha + 2))(\beta_\alpha/2 + \eta)^2 \, n^{(\alpha + 2)/\alpha}}{|z|^2 - x_{1n}^2} \right\}, \tag{4.11}$$

and for all $|z| > x_{1,n+1}$, *we have*

$$|p_n(w_\alpha; z) \, p_{n+1}(w_\alpha; z)|^{-1} \leqslant \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{|z|^{2n+1}} \cdot \exp \left\{ \frac{2K(\alpha, \eta) + 2(\alpha/(\alpha + 2))(\beta_\alpha/2 + \eta)^2 \, (n+1)^{(\alpha + 2)/\alpha}}{|z|^2 - x_{1,n+1}^2} \right\}, \tag{4.12}$$

where $K(\alpha, \eta)$ *as in Lemma* 4.3.

Proof. First, we note that (4.12) is a direct consequence of (4.11) and the fact that $x_{1n} < x_{1,n+1}$. So we give the proof of (4.11) only.

Since w_α is an even weight function, it follows that (see [18, Sect. 2.3(2)])

$$p_n(w_\alpha; z) = \gamma_n z^{n-2[n/2]} \prod_{k=1}^{[n/2]} (z^2 - x_{kn}^2).$$

Hence, for any $z \in \mathbb{C}$, with $|z| > x_{1n}$, we have

$$\begin{aligned}
|p_n(w_\alpha; z)| &= \gamma_n |z|^{n-2[n/2]} \prod_{k=1}^{[n/2]} |z^2 - x_{kn}^2| \\
&= \gamma_n |z|^n \exp\left(\sum_{k=1}^{[n/2]} \log\left|1 - \frac{x_{kn}^2}{z^2}\right|\right) \\
&\geqslant \gamma_n |z|^n \exp\left\{\sum_{k=1}^{[n/2]} \log\left(1 - \frac{x_{kn}^2}{|z|^2}\right)\right\} \\
&\geqslant \gamma_n |z|^n \exp\left(-\sum_{k=1}^{[n/2]} \frac{x_{kn}^2}{|z|^2 - x_{kn}^2}\right),
\end{aligned}$$

and we also have

$$\frac{1}{|z|^2 - x_{kn}^2} \leqslant \frac{1}{|z|^2 - x_{1n}^2}, \qquad \text{for all} \quad k = 1, 2, 3, ..., n,$$

which implies that

$$|p_n(w_\alpha; z)|^{-1} \leqslant \frac{1}{\gamma_n |z|^n} \exp\left(\frac{1}{|z|^2 - x_{1n}^2} \sum_{k=1}^{[n/2]} x_{kn}^2\right).$$

The proof of (4.11) follows by combining (4.6) with this last inequality.

5. Proofs of Theorems 3.1–3.3

The proofs of Theorems 3.1–3.3 start by looking at the error terms that are given in (2.1), (2.2), and (2.3). In each case, we take the absolute value of the error and we estimate the factors that appear on the right-hand side from above. The previous section gives us basically all the estimates that are needed to accomplish this task. Moreover, the same technique that we used in the proof of [1, Theorem 2.1] can be used in proving these theorems. Hence, in view of this brief introduction, we find it appropriate to prove Theorem 3.1 only.

Proof of Theorem 3.1. From Lemma 4.2, it can easily be seen that the proof will be finished if we show that

$$\limsup_{n\to\infty} (|\Delta_n^{(\alpha,0)}|)^{1/n} < 1.$$

To do so, let

$$I_n = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{p_n(w_\alpha; z)\, p_{n+1}(w_\alpha; z)},$$

and choose C_n to be the circle $|z| = R_n$, such that

$$R_n^2 \geqslant \frac{x_{1,n+1}^2}{1-\varepsilon}, \qquad \text{for} \quad a < \varepsilon < 1. \tag{5.1}$$

Using (4.12) with $|z| = R_n$, we conclude that

$$\begin{aligned} &|p_n(w_\alpha; z)\, p_{n+1}(w_\alpha; z)|^{-1} \\ &\quad\leqslant \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R_n^{2n+1}} \\ &\quad\cdot \exp\left\{\frac{2K(\alpha,\eta) + 2(\alpha/(\alpha+2))(\beta_\alpha/2+\eta)^2\,(n+1)^{(\alpha+2)/\alpha}}{\varepsilon R_n^2}\right\}. \end{aligned} \tag{5.2}$$

From the assumption that $\sigma = \limsup_{R\to\infty} (\log M_f(R))/2R^\alpha$, we can find, for any $\delta > 0$, an $N_\delta \in \mathbb{N}$ such that

$$M_f(R_n) \leqslant \exp\{2(\sigma+\delta)\, R_n^\alpha\}, \tag{5.3}$$

for all $R_n \geqslant N_\delta$.

Using (5.2), (4.7), and (5.3), we conclude, for large enough R_n, that

$$\begin{aligned} |I_n| \leqslant{}& A(\alpha,\eta) \left(\frac{\beta_\alpha}{2}+\eta\right)^{2n} (n!)^{2/\alpha} \cdot \frac{1}{R_n^{2n}} \\ &\cdot \exp\left\{2(\sigma+\delta)\, R_n^\alpha + \left(2K(\alpha,n) + 2\left(\frac{\alpha}{\alpha+2}\right)\right.\right. \\ &\left.\left.\cdot \left(\frac{\beta_\alpha}{2}+\eta\right)^2 (n+1)^{(\alpha+2)/\alpha}\right) \Big/ \varepsilon R_n^2\right\}. \end{aligned}$$

Next, we choose an R_n which minimizes the right-hand side of this last inequality, and which, at the same time, satisfies (5.1) for $a < \varepsilon < 1$. So we consider the function

$$h(R) = \frac{1}{R^{2n}} \cdot \exp\left\{2(\sigma+\delta)\, R^{\alpha} + \left(2K(\alpha, \eta) + 2\left(\frac{\alpha}{\alpha+2}\right)\right.\right.$$

$$\left.\left.\cdot\left(\frac{\beta_\alpha}{2}+\eta\right)^2 (n+1)^{(\alpha+2)/\alpha}\right)\Big/\varepsilon R^2\right\}.$$

By differentiating $h(R)$ and setting $h'(R) = 0$, we get

$$2\alpha(\sigma+\delta)\, R^{\alpha+2} - \frac{2}{\varepsilon}\left\{2K(\alpha+\eta) + 2\left(\frac{\alpha}{\alpha+2}\right)\right.$$

$$\left.\times\left(\frac{\beta_\alpha}{2}+\eta\right)^2 (n+1)^{(\alpha+2)/\alpha}\right\} - 2nR^2 = 0. \tag{5.4}$$

Hence, we choose R_n to satisfy (5.4), and from this choice of R_n and (4.5), we obtain that

$$R_n^2 \geqslant \frac{x_{1,n+1}^2}{\{2(\alpha+2)(\sigma+\delta)\,\varepsilon\beta_\alpha^\alpha\}^{2/(\alpha+2)}}.$$

Consequently, (5.1) will be satisfied if

$$\sigma+\delta = \frac{(1-\varepsilon)^{(\alpha+2)/2}}{2(\alpha+2)\,\beta_\alpha^\alpha \varepsilon} < \frac{(1-a)^{(\alpha+2)/2}}{2(\alpha+2)\,\beta_\alpha^\alpha a} = \tau(\alpha). \tag{5.5}$$

From (5.4), we have

$$2(\sigma+\delta)\, R_n^\alpha = \frac{2n}{\alpha} + \left(4K(\alpha, \eta) + 4\left(\frac{\alpha}{\alpha+2}\right)\right.$$

$$\left.\times\left(\frac{\beta_\alpha}{2}+\eta\right)^2 (n+1)^{(\alpha+2)/\alpha}\right)\Big/\alpha\varepsilon R_n^2.$$

Therefore,

$$|I_n| \leqslant A(\alpha, \eta)\left(\frac{\beta_\alpha}{2}+\eta\right)^{2\alpha n/(\alpha+2)}$$

$$\cdot\left[\frac{(\alpha+2)\,\varepsilon(\sigma+\delta)}{2}\right]^{2n/(\alpha+2)}\cdot\left(\frac{n!}{n^n}\right)^{2/\alpha}$$

$$\exp\left\{\frac{2n}{\alpha}+\frac{(4+2\alpha)(\alpha+2)^{2/(\alpha+2)}\,(\sigma+\delta)^{2/(\alpha+2)}}{2^{2/(\alpha+2)}\varepsilon^{\alpha/(\alpha+2)}\alpha(\beta_\alpha/2+\eta)^{4/(\alpha+2)}\,(n+1)^{2/\alpha}}\right.$$

$$\left.\cdot\left[K(\alpha, \eta)+\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{\beta_\alpha}{2}+\eta\right)^2(n+1)^{(\alpha+2)/\alpha}\right]\right\}.$$

By using (5.5) and the Stirling formula, we can rewrite the last inequality, for sufficiently large n, as

$$|I_n| \leqslant A^*(\alpha, \eta)\, n^{1/\alpha}\left\{\left(\frac{1-\varepsilon}{4}\right)(1+T_1(\eta))\right.$$

$$\left.\cdot\exp\left[b_n+\left(\frac{1-\varepsilon}{2\varepsilon}\right)(1+T_2(\eta))\right]\right\}^n,$$

where $A^*(\alpha, \eta)$ is a constant that depends on α and η, $b_n \to 0$ as $n \to \infty$, and $T_i(\eta) \to 0$ as $\eta \to 0$, $i=1, 2$.

Since $g(\varepsilon)=((1-\varepsilon)/4)\exp((1-\varepsilon)/2\varepsilon)$ is a continuous, decreasing function on $(0, 1)$ and $g(a)=1$, it follows that $0<g(\varepsilon)<1$ for $a<\varepsilon<1$. Consequently, we can find, for a small enough η and a large enough n, a number $\rho<1$ such that

$$\frac{(1-\varepsilon)}{4}(1+T_1(\eta))\exp\left\{b_n+\left(\frac{1-\varepsilon}{2\varepsilon}\right)(1+T_2(\eta))\right\}<\rho<1,$$

which establishes that the series $\sum_k |I_k|$ is convergent.

Therefore, for sufficiently large n, we have

$$|\Delta_n^{(\alpha,0)}| \leqslant \sum_{k=n}^{\infty}|I_k| \leqslant M(\alpha, \eta, \varepsilon)\, n\cdot\left\{\frac{(1-\varepsilon)}{4}(1+T_1(\eta))\right.$$

$$\left.\cdot\exp\left[b_n+\left(\frac{1-\varepsilon}{2\varepsilon}\right)(1+T_2(\eta))\right]\right\}^n,$$

where $M(\alpha, \eta, \varepsilon)$ is a constant that depends on α, η, and ε.

Hence,

$$\limsup_{n\to\infty} |\Delta_n^{(\alpha,0)}|^{1/n}$$
$$\leqslant \left\{\frac{(1-\varepsilon)}{4}(1+T_1(\eta))\cdot\exp\left[\left(\frac{1-\varepsilon}{2\varepsilon}\right)(1+T_2(\eta))\right]\right\}.$$

Since η can be chosen as small as we like, and $T_i(\eta)\to 0$ as $\eta\to 0$, it follows that

$$\limsup_{n\to\infty} |\Delta_n^{(\alpha,0)}|^{1/n} \leqslant g(\varepsilon) < 1,$$

which proves Theorem 3.1.

APPENDIX

We conclude this paper by giving a table of the approximate values of $\tau(\alpha)$ for some values of α in [0.1, 10.0] (see Table I), and we also give two graphs of $\tau(\alpha)$ for α in [0.0, 10.0] and [0.0, 30.0] (see Figs. 1 and 2). The table and the graphs were obtained and plotted by Mathematica on a Macintosh PC.

TABLE I

α	$\tau(\alpha)$	α	$\tau(\alpha)$
0.1	0.0726623	5.2	0.199904
0.4	0.20815	5.5	0.190176
0.7	0.274788	5.8	0.180896
1.0	0.306916	6.1	0.172052
1.3	0.320245	6.4	0.163636
1.6	0.322793	6.7	0.155634
1.9	0.31898	7.0	0.148028
2.2	0.311377	7.3	0.140803
2.5	0.301544	7.6	0.133942
2.8	0.290456	7.9	0.127427
3.1	0.278733	8.2	0.121242
3.4	0.26678	8.5	0.115371
3.7	0.254857	8.8	0.109797
4.0	0.243136	9.1	0.104506
4.3	0.231724	9.4	0.099482
4.6	0.220692	9.7	0.0947121
4.9	0.210078	10.0	0.0901826

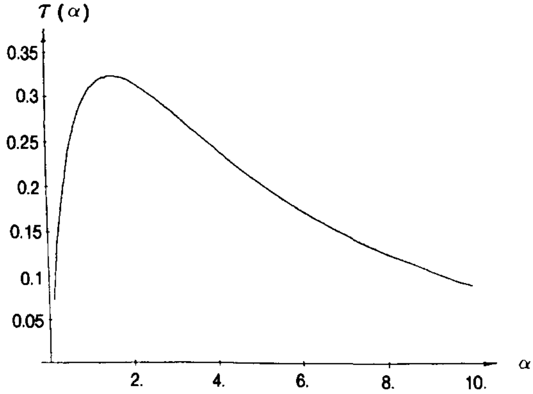

FIGURE 1

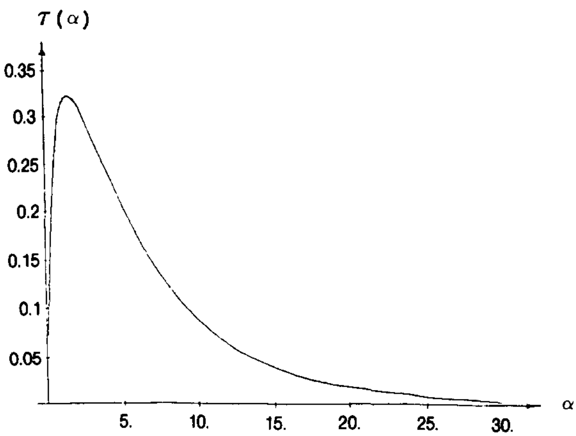

FIGURE 2

ACKNOWLEDGMENTS

We thank the referees for bringing [11] to our attention, and for their useful remarks that helped us in presenting this paper in an up-to-date form.

REFERENCES

1. R. AL-JARRAH, Error estimates for Gauss–Jacobi quadrature formula with weights having the whole real line as their support, *J. Approx. Theory* **30** (1980), 309–314.

2. R. Al-Jarrah, An error estimate for Gauss–Jacobi Quadrature formula with the Hermite weight $w(x) = \exp(-x^2)$ *Publ. Inst. Math. Beograd* (*N.S.*) **33**, No. 47 (1983), 17–22.
3. R. Al-Jarrah, Error estimates for Gauss–Jacobi Quadrature formula with special weights, *Arabian J. Sci. Engrg.* **9** (1984), 33–38.
4. R. Al-Jarrah, On the Lagrange interpolation polynomials of entire functions, *J. Approx. Theory* **41** (1984), 170–178.
5. R. Al-Jarrah, On the Hermite interpolation of entire functions, *Math. Japon.* **33**, No. 2 (1988), 167–174.
6. G. Freud, "Orthogonal Polynomials," Pergamon, Oxford, 1971.
7. G. Freud, Error estimates for Gauss–Jacobi quadrature formulae, *in* "Topics in Numerical Analysis" (J. Miller, Ed.), pp. 113–121, Academic Press, London, 1973.
8. G. Freud, On the greatest zero of orthogonal polynomial, I, *Acta Sci. Math.* **34** (1973), 91–97.
9. A. A. Goncar and E. A. Rahmanov, Equilibrium measure and distribution of zeros of extremal polynomials, *Math. USSR.-Sb.* **53** (1986), 119–130.
10. A. Knopfmacher and D. S. Lubinsky, Mean convergence of Lagrange interpolation for Freud's weights with application to product integration rules, *J. Comput. Appl. Math.* **17** (1989), 79–103.
11. D. S. Lubinsky, Geometric convergence of Lagrangian interpolation and numerical integration rules over unbounded contours and intervals, *J. Approx. Theory* **39** (1983), 338–360.
12. D. S. Lubinsky, H. N. Mhaskar, and E. B. Saff, Freud's conjecture for exponential weights, *Bull. Amer. Math. Soc.* (*N.S.*) **15**, No. 2 (1986), 217–221.
13. D. S. Lubinsky, H. N. Mhaskar, and E. B. Saff, A proof of Freud's conjecture for exponential weights, *Constr. Approx.* **4** (1988), 65–83.
14. H. N. Mhaskar, Weighted polynomial approximation of entire functions, I, *J. Approx. Theory* **35** (1982), 203–213.
15. H. N. Mhaskar, Weighted polynomial approximation of entire functions, II, *J. Approx. Theory* **33** (1981), 59–68.
16. H. N. Mhaskar and E. B. Saff, Where does the sup norm of a weighted polynomial live? *Constr. Approx.* **1** (1985), 71–91.
17. P. Nevai, Mean convergence of Lagrange interpolation, II *J. Approx. Theory* **30** (1980), 263–276.
18. G. Szegő, "Orthogonal Polynomials," 3rd, ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1967.